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The probability distribution of the percolation threshold in a large system

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Abstract. We show that the distribution of the percolation threshold in a large finite system does not converge to a Gaussian when the size of the system goes to infinity, provided that the two widely accepted definitions of correlation length are equivalent. The shape of the distribution is thus directly related to the presence or absence of logarithmic corrections in the power law for the correlation length. The result is obtained by estimating the rate of decay of tail of the limiting distribution in terms of the correlation length exponent ν. All results are rigorously proven in the 2D case. Generalizations for three dimensions are also discussed.

The percolation phenomena have been the subject of many recent theoretical and experimental studies. Apart from their theoretical interest they serve as a guide for the understanding of a wide variety of applied problems such as transport properties of random media, spreading of epidemics and forest fires, statistical tomography and many others. The simplest percolation problem can be formulated as follows. Consider a periodic lattice in d-dimensional space whose bonds (i.e. edges) are occupied or vacant with probabilities p and $1-p$ independent of one another. For a given realization of occupied and vacant bonds, two vertices of the lattice are called connected if they can be joined by a path consisting of occupied bonds. The site percolation problem is defined similarly, but in this case it is the sites that are occupied or vacant, and two sites are connected if they are occupied and one can get from one to another by a path with steps of length one which visits only occupied sites. There are several other percolation models, with different notions of connectedness. In this paper we study short-range percolation models on a d-dimensional lattice, which, for simplicity, we will take to be the cubic lattice.

It is well known (see [11]) that for each $d > 1$ there exists a positive number $p_c < 1$ (its value depends on d and the notion of connectedness) such that for $p < p_c$, there are no infinite connected sets, while for $p > p_c$ a unique connected set (also called the infinite cluster) exists. p_c is called the percolation threshold and is the value of the density of occupied bonds or sites at which the second-order percolation transition occurs. Since the percolation threshold $p_c^{(L)}$ in a large system of linear size L (defined below) is a function of L^d independent random variables (bonds or sites) one may ask whether the analogues of the law of large numbers (LLN) and the central limit theorem (CLT) hold. We show that this is indeed true for the LLN, namely as $L \to \infty$, $p_c^{(L)}$ converges to the non-random number p_c. One may expect that this analogy with classical probability results extends to CLT, i.e.

that the distribution of $p_c^{(L)}$ normalized by its standard deviation converges to the Gaussian. Such claims (based on numerical simulations) have indeed been made in the literature [9] (see also [22]).

In this paper we present conditions under which the distribution of $p_c^{(L)}$ does not become Gaussian in the $L \to \infty$ limit. We note that knowledge of the limiting distribution is crucial for the proper interpretation of numerical estimates of the critical density, obtained from numerical simulations. Indeed, the decay of the tail of the limiting distribution determines the confidence interval for such data. We present the conditions under which the tail decays *slower* than that in the Gaussian case, which implies that the interval will be wider, i.e. the numerical data will be more spread around the actual value of p_c and, consequently, the data have to be interpreted with more care than in the Gaussian case.

An alternative view on the percolation transition, which is especially useful in the study of two-dimensional models is provided by the crossing probability defined as the probability $\pi_L(p)$ that there exists a connected path linking two given opposite faces of the cube of side length L, centred at the origin. If $p < p_c$ then $\pi_L(p) \to 0$ as $L \to \infty$ exponentially fast, i.e.

$$-\lim_{L\to\infty} \frac{1}{L}\log \pi_L(p) = \xi^{-1}(p). \tag{1}$$

If $p > p_c$ then $1 - \pi_L(p) \to 0$ as $L \to \infty$. This convergence is known to be exponential in all dimensions [6, 12]. The function $\xi(p)$ is a widely used definition of the correlation length, which is a central quantity in the study of the percolation transition. There is another, very convenient definition (see [4]): fix a number $\epsilon > 0$; for $p < p_c$, let $L_0 = L_0(p, \epsilon)$ be the smallest L for which $\pi_L(p) < \epsilon$; for $p > p_c$, let L_0 be the smallest L for which $\pi_L(p) > 1 - \epsilon$. Even though L_0 depends on the choice of ϵ, it was shown in [17] that for two different values of ϵ, $L_0(p, \epsilon_1) \asymp L_0(p, \epsilon_2)$, that is the ratio $L_0(p, \epsilon_1)/L_0(p, \epsilon_2)$ is bounded away from 0 and ∞ in a neighbourhood of p_c for any short-range two-dimensional model. It was also proven in [17] that in two dimensions $L_0(p_c - y, \epsilon) \asymp L_0(p_c + y, \epsilon)$ as $y \downarrow 0$. For $d > 2$ there are no rigorous results but, according to the usual scaling assumption, as $p \to p_c$ from above or from below the correlation length should behave as $(p-p_c)^{\nu_+}$ or as $(p-p_c)^{\nu_-}$, respectively. The values of $\nu_\pm$ have been calculated numerically for various models [22, 16] and found to depend only on the dimensionality of the system; moreover $\nu_+ = \nu_-$ with high accuracy (note that in two dimensions this follows from the above relation $L_0(p_c-y, \epsilon) \asymp L_0(p_c+y, \epsilon)$). However, one cannot exclude the presence of logarithmic corrections in the asymptotics of the correlation length as $p \to p_c$. Logarithmic corrections, indeed, appear in the theory of critical phenomena (see [18]). While they are usually present at the upper critical dimension of the studied system, there are no numerical, experimental or theoretical grounds to rule them out here. In fact, it is not even known whether $\xi(p) \asymp L_0(p)$. As we will see, if ξ and L_0 are asymptotically equivalent, then the (normalized) random variables $p_c^{(L)}$ cannot have a Gaussian limit. The following procedure is used to numerically find the value of p_c for a given percolation model (see, for example [7, 22]). We will describe it for the bond model to illustrate the ideas; the analogues for the other models are obvious. Take a cube of side length L with all bonds occupied and start deleting them at random until the connection between a fixed pair of opposite faces is broken. Calculate the fraction of the remaining bonds (i.e. their number divided by the number of all bonds in the cube) and denote this fraction by $p_c^{(L)}$. This is our finite-volume percolation threshold. It is a random variable since it depends on the realization of the bond deleting process (i.e. in computer experiments, on the generated sequence of random numbers). Since a connection is highly improbable for $p < p_c$ and highly probable for $p > p_c$, the fraction of bonds remaining when the connection has just been broken should be approximately equal to p_c. It can indeed be rigorously shown that $p_c^{(L)} \to p_c$ as $L \to \infty$ in probability, i.e. for any $\delta > 0$

$$\mathrm{Prob}(|p_c^{(L)} - p_c| > \delta) \to 0 \qquad L \to \infty. \tag{2}$$

This is done in [3] by an argument which applies to any short-range percolation model in an arbitrary number of dimensions ($\geqslant 2$). The nature of the fluctuations of $p_c^{(L)} - p_c$ is a much more subtle question which we now discuss. We are first going to find the asymptotic behaviour (scaling) of the second moment of $p_c^{(L)} - p_c$ as $L \to \infty$. The idea is to use the well known formula [8]

$$E(X^2) = \int_0^\infty 2yP(|X| \geqslant y)\,\mathrm{d}y \tag{3}$$

which expresses the second moment of a random variable in terms of its distribution function. We emphasize that the formula is true for arbitrary, possibly discrete-valued, random variables. Applying this formula with $X = p_c^{(L)} - p_c$, it is clear that lower and upper bounds of the same order on $P(|p_c^{(L)} - p_c| \geqslant y)$ imply bounds on $E((p_c^{(L)} - p_c)^2)$ differing just by a multiplicative constant, independent of L, i.e. they determine the scaling of the second moment. Now, $P(|p_c^{(L)} - p_c| \geqslant y) = P(p_c^{(L)} \geqslant p_c + y) + P(p_c^{(L)} < p_c - y)$. To understand the nature of the above quantities let us look, for example, at $P(p_c^{(L)} < p_c - y)$. This is the probability that, when removing bonds at random, a left-to-right (L–R) connection persists, even when the fraction of the remaining bonds is $p_c - y$. It is therefore quite natural that

$$P(p_c^{(L)} < p_c - y) \approx \pi_L(p_c - y) \qquad \text{as } L \to \infty \tag{4}$$

and similarly

$$P(p_c^{(L)} > p_c - y) \approx 1 - \pi_L(p_c + y) \qquad \text{as } L \to \infty. \tag{5}$$

We emphasize that the probabilities on the left-hand sides of these approximate inequalities are to be understood in the sense of randomly deleted consecutive bonds, while on the right-hand sides they are calculated in the percolation model with density $p_c \pm y$. These approximations can indeed be precisely formulated and rigorously justified. We refer the reader to [3] for details, which lead to the following second moment bounds

$$\begin{aligned} c\left[\int_0^{p_c} y\pi_L(p_c - y)\,\mathrm{d}y + \int_0^{1-p_c} y\pi_L(p_c + y)\,\mathrm{d}y\right] &\leqslant E((p_c^{(L)} - p_c)^2) \\ &\leqslant C\left[\int_0^{1-p_c} y\pi_L(p_c + y)\,\mathrm{d}y + \int_0^{p_c} y\pi_L(p_c + y)\,\mathrm{d}y\right] \end{aligned} \tag{6}$$

with the constants $0 < c < C < \infty$ independent of L. This suggests a connection between the second moment bound and the correlation length L_0. Although formula (6) is true in any dimension, we have control over $\pi_L(p)$ only in two dimensions so our result about the second moment of $p_c^{(L)} - p_c$ is limited to this case. The lower bound is obtained by simply disregarding the second integral on the left-hand side of 6 and restricting the first integral to the interval $(0, y_0(L))$, where y_0 is the function, inverse to L_0, i.e. $L_0(y_0(L)) = L$ (we are henceforth fixing a value of ϵ and suppressing dependence on ϵ in the notation). This means that we are taking contributions only from those y for which L is smaller than the correlation length $L_0(p_c - y)$. For y in this interval $\pi_L(p) \geqslant \epsilon$, and we obtain the lower bound

$$E((p_c^{(L)} - p_c)^2) \geqslant c\epsilon \int_0^{y_0(L)} y\,\mathrm{d}y = \tfrac{1}{2}c\epsilon y_0(L)^2. \tag{7}$$

Note that this bound holds in any dimension. To get a similar upper bound let us first consider $\int_0^{p_c} y\pi_L(p_c - y)\,dy = \int_0^{y_0(L)} y\pi_L(p_c - y)\,dy + \int_{y_0(L)}^{p_c} y\pi_L(p_c - y)\,dy$. The first integral is, of course, bounded by $\frac{1}{2}y_0^2(L)$ (because $\pi_L(p_c - y) \leqslant 1$). To estimate the second integral, we use the rescaling lemma [1], which says, in essence, that in the 2D case if for some L_0, $\pi_{L_0}(p_c - y) \leqslant \epsilon$, where $\epsilon < 1$ is a suitable constant, then for $L \geqslant L_0$, $\pi_L(p_c - y) \leqslant \pi_{L_0}(p_c - y)^{c(L/L_0)} \leqslant \exp[-c|\log\epsilon|L/L_0]$ with c independent of L and p. A fundamental statement of the scaling theory is that as $y \to 0$, $L_0(p_c - y)$ behaves as $y^{-\nu}$, perhaps with a logarithmic correction. By a direct asymptotic calculation one can check that in both cases (with or without the correction),

$$\int_{y_0(L)}^{p_c} y\pi_L(p_c - y)\,dy \leqslant Cy_0(L)^2. \tag{8}$$

More generally, the above bound was verified in [3] for a large class of functions L_0 of the form of the pure power multiplied by slowly varying functions ([10], ch 8.8) which grow slower than any power and thus include logarithmic corrections. This completes the estimate of the first term on the right-hand side of (6). The second term is estimated exactly in the same way, using the above-mentioned equivalence of $L_0(p_c + y)$ and $L_0(p_c - y)$. The conclusion of the above is

$$cy_0(L)^2 \leqslant E((p_c^{(L)} - p_c)^2) \leqslant Cy_0(L)^2 \tag{9}$$

with two positive constants c and C, independent of L. Let us note that if $L_0(p_c - y)$ and $L_0(p_c + y)$ are both equal to $y^{-\nu}$ up to a multiplicative constant, then $y_0(L)^2 = L^{-2/\nu}$ and we recover the expression given in [9, 22] for the $\mathrm{Var}(p_c^{(L)}) = E[(p_c^{(L)} - E(p_c^{(L)})^2]$. If we normalize $p_c^{(L)} - p_c$ by $y_0(L)$, we thus obtain a sequence of random variables with bounded second moments. This brings us to the question of the limiting distribution of these normalized variables. We remark here that in the spirit of the classical central limit theorem it is slightly more natural to study the limiting distribution of $p_c^{(L)} - E(p_c^{(L)})$ rather than $p_c^{(L)} - p_c$. Although in the two dimensions bond model for a square box they are exactly the same because of self-duality [11], in general they are different. For technical reasons we shall first study $p_c^{(L)} - p_c$ and then return to $p_c^{(L)} - E(p_c^{(L)})$. It is shown in [3] that the sequence (or at least subsequence) $(p_c^{(L)} - p_c)/y_0(L)$ indeed converges to a limiting random variable Y which is not constant (i.e. the limiting distribution is not trivial). We want to study the probability $P(Y \geqslant y)$ for large y, in order to gain information about the tails of the distribution of Y. To this end, let us use the correlation length ξ introduced above. Its definition roughly expresses the fact that as $L \to \infty$, $\pi_L(p)$ (or $1 - \pi_L(p)$) decays exponentially with the rate $1/\xi(p)$. This does not imply that $\pi_L(p)$ behaves like $e^{-L/\xi(p)}$ but inequality in one direction is known in 2D. Namely, there exists a positive constant, independent of L and p, such that

$$\pi_L(p) \geqslant Ke^{-L/\xi(p)} \qquad \text{for } p \leqslant p_c \tag{10}$$

and similarly

$$1 - \pi_L(p) \geqslant Ke^{-L/\xi(p)} \qquad \text{for } p \geqslant p_c \tag{11}$$

(we owe this bound to L Chayes, see [3] for details). It is also known [4] that for $p < p_c$

$$\xi(p) \leqslant L_0(p) \leqslant \left(a + b\log\frac{1}{p_c - p}\right)\xi(p) \tag{12}$$

for some constants a and b, independent on p, where L_0 is the correlation length introduced earlier. It is natural to ask whether the upper bound in the above inequality can be strengthened to the form constant $\xi(p)$, i.e. whether one can remove the logarithmic correction. This is a difficult open question, which is physically relevant. In particular, we will show that the absence of the logarithmic correction implies that the distribution of $p_c^{(L)} - p_c)/y_0(L)$ does not converge to a Gaussian as $L \to \infty$. For any L we have

$$P\left(\frac{p_c^{(L)} - p_c}{y_0(L)} \leqslant -y\right) = P(p_c^{(L)} - p_c \leqslant -y_0(L)y)$$
$$\approx \pi_L(p_c - y_0(L)y) \geqslant K \exp[-L/\xi(p_c - y_0(L)y)] \tag{13}$$

and, if $\xi(p) \asymp L_0(p)$, the right-hand side is bounded below by $K\mathrm{e}^{-\text{constant } y^{\nu}}$, possibly with a logarithmic correction in the exponent. Now, the numerical value of ν in 2D is close to $\frac{4}{3}$ and the value $\frac{4}{3}$ was also obtained by conformal field theory calculations [2]. This would imply that the tail of the limiting distribution decays slower than that of a Gaussian. Alternatively, one can obtain from inequality (13) a lower bound on the moment generating function of the limiting distribution Y:

$$E(\mathrm{e}^{tY}) \geqslant K \exp[\text{constant } t^{(\nu/\nu-1)}] \tag{14}$$

which excludes a Gaussian since $\nu < 2$ and for a Gaussian variable Γ

$$E(\mathrm{e}^{t\Gamma}) \sim \exp[\text{constant } t^2] \qquad \text{as } t \to \infty. \tag{15}$$

Finally we remark that the same analysis with some minor technical modifications applies to the 'natural' physical variable $P_L = [p_c^{(L)} - E(p_c^{(L)})]/a_L$, where a_L is a proper normalization, for example $a_L = \surd(\mathrm{Var}(p_c^{(L)} - E(p_c^{(L)})))$. To summarize the above argument: equivalence of the two definitions of correlation length implies that the limiting distribution of the finite-volume percolation threshold is not Gaussian and provides an estimate (14) on the tail of the distribution. Or equivalently, if the limiting distribution is Gaussian, then the two correlation lengths have to differ by a divergent factor. The Gaussian behaviour was indeed reported in [9] (as shown in [3] it also follows from statements in [22]). At present we are unable to provide a rigorous answer to the question of equivalence of L_0 and ξ. Most of the above applies, from the rigorous point of view, only to 2D models, since it depends on several (highly non-trivial) results which so far have only been proven in two dimensions. A generalization of the above results to $d > 2$ is, however, so interesting that it deserves mentioning.

(i) The 'law of large numbers' carries over to any d, together with exponential bounds on the rate of convergence (large deviation probabilities), the form of which depends now on whether we are above or below p_c. More precisely, up to power-law corrections in L we have

$$P(p_c^{(L)} < p_c - \epsilon) \approx \exp\left(-cL/\xi\left(p_c - \frac{\epsilon}{2}\right)\right) \tag{16}$$

and

$$P(p_c^{(L)} > p_c + \epsilon) \approx \exp(-\kappa(\epsilon)L^{d-1}). \tag{17}$$

Proofs of these statements rely on analogous bounds for the crossing probabilities π_L. While the first bound follows from standard estimates in the subcritical region (see e.g. [11]), the second bound is non-trivial [6] (see also [23])) and at the time of writing there seem to be no rigorous results about the behaviour of $\kappa(\epsilon)$ as $\epsilon \to 0$. We shall however, explore the consequences of the physically plausible assumption that $\kappa(\epsilon)$ behaves as constant$(\xi(p_c+\epsilon))^{1-d}$ (an expression that depends only on the correlation length), which makes the quantity $\kappa(\epsilon)L^{d-1}$ dimensionless and depends only on the correlation length ξ.

(ii) If we assume in addition (in agreement with numerical results) that

$$\nu_+ = \nu_- = \nu \tag{18}$$

then we recover the previous asymptotics

$$E((p_c^{(L)} - p_c)^2) \asymp y_0(L)^2 \tag{19}$$

provided that

$$\nu > \frac{2}{d}. \tag{20}$$

It is interesting that this condition, which emerges from a careful repetition of the estimates leading to (7), is identical to the Harris criterion for the relevance of disorder in random ferromagnets [5, 15]. As shown in [5, 19], the weaker condition $\nu \geqslant 2/d$ is satisfied in any dimension. For the short-range models considered here, numerical simulations clearly indicate that the sharp inequality holds.

(iii) The bound

$$P\left(\frac{p_c^{(L)} - p_c}{y_0(L)} \geqslant z\right) \geqslant e^{-cz^\nu} \tag{21}$$

in d dimensions would mean that the limiting distribution of $(p_c^{(L)} - p_c)/y_0(L)$ is even more radically non-Gaussian than in two dimensions. The reason is that, for $d > 2$, ν is less than 1 and therefore the tail of the limiting distribution decays so slowly that its moment generating function is infinite. Its moments, μ_k, can be bounded below by a calculation analogous to (7) and it turns out that they grow so fast that they do not determine the distribution uniquely [20, 21]. As d grows, ν decreases and the tail of the limiting distribution of the finite volume percolation threshold becomes bigger. According to the well known prediction $\nu = \frac{1}{2}$ for $d \geqslant 6$. This was in fact proven for large d in [14]. Consequently, in contrast to dealing with extensive random variables (like energy or magnetization), the distribution of $(p_c^{(L)} - p_c)/y_0(L)$ becomes extremely non-Gaussian above the upper critical dimension.

(iv) When $d > 2$, due to the different forms of the large deviation bounds (16) and (17) below and above p_c, it is possible that the limiting distribution of $(p_c^{(L)} - p_c)/y_0(L)$ is not symmetric. Indeed, it is not even known whether in this case the limit $\lim_{L\to\infty} \pi_L(p_c)$ lies strictly between 0 and 1.

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